MATH 220.204, MARCH 29 2019

1. (2018 WT2 Final Exam) For each example below, determine whether $\mathcal{R}$ is a function from A to B.
   - $A = \mathbb{R}, B = \mathbb{Z}$, $\mathcal{R} = \{(x, y) \in \mathbb{R} \times \mathbb{Z} : x = 3y + 1\}$
     It is not a function, for example when $x = \sqrt{2}$ there is no associated integer y such that $x = 3y + 1$.
   - $A = \mathbb{Q}_{\geq 0}, B = \mathbb{R}$, $\mathcal{R} = \{(x, y) \in \mathbb{Q}_{\geq 0} \times \mathbb{R} : x^2 = y\}$
     It is a function, because for every positive rational number x, the number x^2 is a real number.

2. (2018 WT2 Final) Consider the relation on $\mathbb{Q}$ defined by $a\mathcal{R}b \iff a - b \in \mathbb{Z}$.
   (a) Prove that $\mathcal{R}$ is an equivalence relation.
   - For every a, $a - a = 0 \in \mathbb{Z}$.
   - For every a, b such that $a - b \in \mathbb{Z}$, we have $b - a = -(a - b)$ and so $b - a \in \mathbb{Z}$.
   - For every a, b, c such that $a - b \in \mathbb{Z}$ and $b - c \in \mathbb{Z}$, we have $a - c = (a - b) + (b - c)$ and so $a - c \in \mathbb{Z}$.

   (b) Prove that the following statement is false:
   $$\forall a, b \in \mathbb{Q}, (a\mathcal{R}b \implies (\forall q \in \mathbb{Q}, (qa)\mathcal{R}(qb))).$$
   We must show that there exist a, b such that $a - b \in \mathbb{Z}$ and $\exists q \in \mathbb{Q}, qa - qb \notin \mathbb{Z}$. Take $a = 1, b = 0$, and $q = 1/2$.

   (c) Prove if $a, b \in \mathbb{Q}$ satisfy the property that $\forall q \in \mathbb{Q}, (qa)\mathcal{R}(qb)$, then $a = b$.
   We prove the contrapositive, namely if $a \neq b$ then there exists $q \in \mathbb{Q}$ such that $qa - qb \notin \mathbb{Z}$. If $a \neq b$, then $a - b$ is a nonzero rational number. Write $x = a - b$. Then if we let $q = \frac{1}{2x}$, we have that $qa - qb = \frac{1}{2} \notin \mathbb{Z}$.

3. Let A, B be nonempty sets. Prove that if $|A| \leq |B|$ then $|\mathcal{P}(A)| \leq |\mathcal{P}(B)|$.
   If $|A| \leq |B|$, then there is an injective function $f : A \to B$. Define a function $F : \mathcal{P}(A) \to \mathcal{P}(B)$ by, for every $S \subseteq A$, $F(S) := f(S) \subseteq B$.
   I claim that F is injective. Suppose that S_1, S_2 are two distinct subsets of A. Then there is some element $x \in A$ which is in one and not the other. WLOG, $x \in S_1$ and $x \notin S_2$. Then $f(x) \in f(S_1)$. For every $y \in S_2$, we have $x \neq y \implies f(x) \neq f(y)$, and thus it follows that $f(x) \notin f(S_2)$. Thus, $f(S_1) \neq f(S_2)$. This proves that F is injective, and thus $|\mathcal{P}(A)| \leq |\mathcal{P}(B)|$.

4. Let $a, b \in \mathbb{Z}$ be integers such that $a^2 - 3ab + b^2 = 0$. Prove that $a = b = 0$. (Hint: Try mod 3.)
   Suppose for a contradiction that at least one of a, b is nonzero. Then they have a greatest common divisor d, and write $a = dx$, $b = dy$. Then $a^2 - 3ab + b^2 = d^2(x^2 - 3xy + y^2)$. Therefore, $x^2 - 3xy + y^2 = 0$, and $\gcd(x, y) = 1$. Now observe

that
$$x^2 - 3xy + y^2 \equiv x^2 + y^2 \pmod 3$$
and thus, $x^2 + y^2 \equiv 0 \pmod 3$. However, observe that for any integer $n \in \mathbb{Z}$,
$$n \equiv 0 \pmod 3 \implies n^2 \equiv 0 \pmod 3$$
$$n \equiv 1 \pmod 3 \implies n^2 \equiv 1 \pmod 3$$
$$n \equiv 2 \pmod 3 \implies n^2 \equiv 1 \pmod 3$$
It thus follows that $x \equiv y \equiv 0 \pmod 3$. But this is impossible, because $\gcd(x, y) = 1$! We have thus reached a contradiction, and so no such integers a, b exist.

5. In this question, you will construct an explicit bijection to prove that the sets
$$\mathcal{P}(\mathbb{N}) = \{S : S \subseteq \mathbb{N}\} \quad \text{and} \quad (0, 1] = \{x \in \mathbb{R} : 0 < x \leq 1\}$$
have the same cardinality. You can prove each step separately, so you may work on later parts first if you prefer.

(a) Let $\mathcal{F} \subset \mathcal{P}(\mathbb{N})$ denote the set of *finite* subsets of $\mathbb{N}$. That is,
$$\mathcal{F} = \{S \subset \mathbb{N} : S \text{ is finite}\}.$$
Prove that $\mathcal{F}$ is countable.

For each $n \geq 0$, let $\mathcal{F}_n = \{S \subset \mathbb{N} : |S| = n\}$. It is obvious that $\mathcal{F}_0 = \{\emptyset\}$ is countable. For every natural number $m \geq n$, let $\mathcal{F}_n(m) = \{S \subset \{1, 2, \dots, m\} : |S| = n\}$. The set $\mathcal{F}_n$ can be written as a denumerable union of finite sets $\mathcal{F}_n = \bigcup_{m=n}^{\infty} \mathcal{F}_n(m)$ and therefore, $\mathcal{F}_n$ is countable for every $n \geq 1$. A denumerable union of countable sets is countable (because $\mathbb{N} \times \mathbb{N}$ is countable), and therefore because $\mathcal{F} = \bigcup_{n=0}^{\infty}$, it follows that $\mathcal{F}$ is countable.

(b) Let $\mathcal{I} = \mathcal{P}(\mathbb{N}) - \mathcal{F}$ be the complement of $\mathcal{F}$. Use the previous part to prove that $\mathcal{I}$ and $\mathcal{P}(\mathbb{N})$ have the same cardinality.

It is obvious that $\mathcal{I}$ is infinite (I'll leave it to you to prove that if you wish). Let $S \subseteq \mathcal{I}$ be any denumerable subset of $\mathcal{I}$. Then there is a bijection from S to $S \cup \mathcal{F}$, because $\mathcal{F}$ is countable. Thus, there is a bijection from $\mathcal{I} = (\mathcal{I} - S) \cup S$ to $\mathcal{P}(\mathbb{N}) = (\mathcal{I} - S) \cup S \cup \mathcal{F}$.

(c) Let $x \in (0, 1]$. Define a sequence of positive integers $a_1 < a_2 < a_3 < \dots$ as follows. For every $n \in \mathbb{N}$, a_n is the smallest positive integer such that
$$\frac{1}{2^{a_n}} < x - \frac{1}{2^{a_1}} - \frac{1}{2^{a_2}} - \cdots - \frac{1}{2^{a_{n-1}}}.$$
Prove that this construction is well-defined. That is, prove that for every n, there will always be such a positive integer a_n, and also prove that $a_1 < a_2 < a_3 < \dots$. Conclude that the above procedure defines a function
$$f : (0, 1] \to \mathcal{I}, \quad f(x) = \{a_1, a_2, a_3, \dots\}.$$

We must show by induction on n that if the above procedure has produced positive integers $a_1 < a_2 < \cdots < a_{n-1}$, then the definition will give a positive integer a_n such that $a_{n-1} < a_n$.
Consider the base case $n = 1$. If $x \in (0, 1]$, then there is at least one positive integer a such that $2^a x > 1$. It then follows that $\frac{1}{2^a} < x$. The set of positive integers a satisfying this property is nonempty, so by WOP, there is a unique minimal such positive integer, which is a_1.
Now we prove the inductive step. Suppose that we have our positive integers $a_1 < a_2 < \cdots < a_{n-1}$. By the definition of a_{n-1}, we have

$$0 < x - \frac{1}{2^{a_1}} - \frac{1}{2^{a_2}} - \cdots - \frac{1}{2^{a_{n-1}}}$$

and therefore by the same reasoning as the base case, there is a unique minimal positive integer a_n such that

$$\frac{1}{2^{a_n}} < x - \frac{1}{2^{a_1}} - \frac{1}{2^{a_2}} - \cdots - \frac{1}{2^{a_{n-1}}}.$$

All that remains is to prove that $a_n > a_{n-1}$. Suppose for a contradiction that $a_n \le a_{n-1}$. Then

$$\begin{aligned} x - \frac{1}{2^{a_1}} - \frac{1}{2^{a_2}} - \cdots - \frac{1}{2^{a_{n-2}}} &> \frac{1}{2^{a_{n-1}}} + \frac{1}{2^{a_n}} \\ &\ge \frac{1}{2^{a_{n-1}}} + \frac{1}{2^{a_{n-1}}} \\ &= \frac{1}{2^{a_{n-1}-1}} \end{aligned}$$

But this contradicts the minimality of a_{n-1}! Therefore, we must have $a_n > a_{n-1}$, as desired.

(d) Prove that if $f(x) = \{a_1, a_2, a_3, \ldots\}$ with $a_1 < a_2 < a_3 < \ldots$, then

$$x = \frac{1}{2^{a_1}} + \frac{1}{2^{a_2}} + \frac{1}{2^{a_3}} + \ldots.$$

Deduce that f is injective.

First, observe that by the minimality of a_n, we have $x - \sum_{i=1}^{n} \frac{1}{2^{a_n}} < \frac{1}{2^{a_n}}$. Also observe that since $1 \le a_1 < a_2 < \cdots < a_n$, we have that $n \ge a_n$ for every n. Thus, it follows that

$$x - \sum_{i=1}^{n} \frac{1}{2^{a_n}} < \frac{1}{2^n}.$$

We need to prove that for each $\epsilon > 0$, there exists some positive integer N such that if $n > N$, then $x - \sum_{i=1}^{n} \frac{1}{2^{a_n}} < \epsilon$. Let us pick N to be any positive integer such that $\frac{1}{2^N} < \epsilon$. Then if $n > N$, we have

$$x - \sum_{i=1}^{n} \frac{1}{2^{a_i}} < \frac{1}{2^n} < \frac{1}{2^N} < \epsilon.$$

as desired.

Injective: Suppose that $x, y \in (0, 1]$ and

$$f(x) = f(y) = \{a_1, a_2, a_3, \ldots\}, \qquad a_1 < a_2 < a_3 < \cdots.$$

Then by the previous part, we have

$$x = \frac{1}{2^{a_1}} + \frac{1}{2^{a_2}} + \frac{1}{2^{a_3}} + \ldots = y.$$

so $x = y$.

(e) Prove that if $x \in (0, 1]$ and $x = \frac{1}{2^{a_1}} + \frac{1}{2^{a_2}} + \frac{1}{2^{a_3}} + \ldots$ for some positive integers $a_1 < a_2 < a_3 < \cdots$, then $f(x) = \{a_1, a_2, a_3, \ldots\}$. Deduce that f is surjective.

Suppose that $f(x) = \{b_1, b_2, b_3, \ldots\}$ for positive integers $b_1 < b_2 < b_3 < \cdots$. We will show by induction on n that $b_n = a_n$. The following argument to prove the inductive step works for the base case $n = 1$. Suppose that $b_i = a_i$ for $i = 1, 2, \ldots, n-1$. We have that

$$x - \sum_{i=1}^{n-1} \frac{1}{2^{a_i}} = \frac{1}{2^{a_n}} + \frac{1}{2^{a_{n+1}}} + \frac{1}{2^{a_{n+2}}} + \cdots > \frac{1}{2^{a_n}}$$

We also have

$$x - \sum_{i=1}^{n-1} \frac{1}{2^{a_i}} = \frac{1}{2^{a_n}} + \frac{1}{2^{a_{n+1}}} + \frac{1}{2^{a_{n+2}}} + \leq \frac{1}{2^{a_n}} + \frac{1}{2^{a_n} \cdot 2} + \frac{1}{2^{a_n} \cdot 4} + \ldots = \frac{1}{2^{a_n - 1}}$$

Therefore, a_n is the minimal positive integer such that $x - \sum_{i=1}^{n-1} \frac{1}{2^{a_i}} > \frac{1}{2^{a_n}}$ and so $b_n = a_n$, as desired.

Surjective: Let $\{a_1, a_2, a_3, \ldots\}$ be any infinite subset of $\mathbb{N}$ written so that $a_1 < a_2 < a_3 < \cdots$. Then take $x = \sum_{i=1}^{\infty} \frac{1}{2^{a_i}}$. We then have $f(x) = \{a_1, a_2, a_3, \ldots\}$.